AAS 10-245

MINIMUM ΔV LAUNCH WINDOWS FOR A FICTIVE POST-2029 APOPHIS DEFLECTION/DISRUPTION MISSION

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In this paper a fictional scenario is considered in which asteroid Apophis passes through a keyhole on April 13th, 2029 and is on a collision course with the Earth. A software package is developed to determine all the feasible launch dates that are compatible with an Interplanetary Ballistic Missile system, which is capable of delivering up to 1500 kg nuclear explosives, with a total ΔV of 4 km/s from a geostationary transfer orbit. Mission analysis for late launch dates, allowing interception or rendezvous for a nuclear subsurface explosion at least 15 days prior to Earth impact on April 13th, 2036 is also performed.

INTRODUCTION

Asteroids and comets have collided with the Earth in the past and are predicted to do so in the future. These collisions have had a significant role in shaping Earth's biological and geological history, most notably the extinction of the dinosaurs. Even in the recent past, near-Earth objects (NEOs) have collided with the Earth. Most of these collisions are with small relatively harmless NEOs. One notable NEO impact was the Tunguska event that occured in Siberia in 1908. This impact has been estimated to have released an explosion energy equivalent to 3-5 megatons of TNT. While this event was relatively harmless, it would be devastating if it occured in a populated area. For this reason, methods and mission architectures capable of deflecting or disrupting such hazardous NEOs need to be developed. A preliminary design of an Interplanetary Ballistic Missile (IPBM) system architecture has previously been conducted.[1] The purpose of this paper is to determine the feasibility of using this IPBM system architecture to deflect or disrupt an NEO that is on a collision course, but with short warning time.

A recent study has concluded that it may be feasible to significantly reduce the impact damage from an Earth-impacting NEO using a nuclear subsurface explosion as late at 15 days prior to impact.[2] This is of course only for relatively small NEOs with a diameter of at most a few hundred meters. For a mitigation mission within a few weeks prior to impact requires a nuclear subsurface explosion. In this paper the asteroid 99942 Apophis will be used as a reference asteroid. The impact probability of Apophis for April 13th, 2036, is currently estimated as approximately 1 in 250,000. However, in this paper, Apophis is assumed to be on a collision course after the Earth-Apophis close encounter on April 13th, 2029. The orbital elements used for this hypothetical case are shown in Table 1(a),[3] while the physical parameters of Apophis are summarized in Table 1(b).

The proposed IPBM system[1] basically consists of a launch vehicle (LV) and an integrated space vehicle (ISV). The ISV consists of an orbital transfer vehicle and a terminal maneuvering vehicle carrying nuclear payloads. A Delta IV Heavy launch vehicle can be chosen as a baseline LV of a primary IPBM system for delivering a 1500-kg (mass) nuclear explosive for a rendezvous mission with a target NEO. The primary IPBM system is capable of a ΔV of at least 3.5 km/s. A secondary IPBM system employs a Delta II class launch vehicle (or a Taurus II) with a smaller ISV carrying a 500-kg nuclear explosive. Basically, a primary

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IPBM system can deliver a 1.5-MT nuclear payload while a secondary system can deliver a 500-KT nuclear payload. A kilotons (KT) of TNT equals to an energy of 4.18×10^{12} Joules.

It will be shown that for the fictional post-2029 Apophis mission, the IPBM system is capable of both rendezvous and interception mission, with several possible launch dates. Included in these launch windows is a "last minute" mission, capable of intercepting Apophis 1-2 months prior to impact. The focus of this paper is on the mission analysis and the development of the software needed to complete the mission analysis. It is assumed that the maximum ΔV limit used when searching for the launch windows is in the 3.5-4 km/s range, corresponding to the IPBM system.[1]

Table 1: Orbital and physical data used for a fictive post-2029 asteroid Apophis mission.

(a) Orbital elements

Elements	Value
epoch MJD	64699
a, AU	1.108243
e	0.190763
i, deg	2.166
Ω, deg	70.23
ω, deg	203.523
M_0, deg	227.857

(b) Physical parameters

Physical Parameter	Value
Rotational Period (hr)	30.5
Mass (kg)	2.1E+10
Diameter (m)	270
H	19.7
Albedo	0.33

In addition to evaluating a mission concept to Apophis after 2029, the purpose of this paper is to outline the methods used to create a software package in Matlab capable of searching for trajectories and launch windows, requiring only ephemeris data of the desired destination. An outline of methods used for the Lambert solver and several of the main functions required can be found in Appendices A and B. This software currently includes the evaluation of interception and rendezvous missions, as well as the ability to search for crewed mission launch dates.[?] Future versions of the software will include the ability to search multiple revolution orbits (multiple-revolution Lambert solutions), multiple gravity assists, resonant orbits, and the ability to search for phasing orbits. This paper is essentially a case study to show the capabilities of the software that has been developed. Also the software has been validated by reproducing graphs and trajectories for a mission to Apophis for a radio tagging mission.[4]

MISSION ANALYSIS AND DESIGN

In this section both the rendezvous and interception missions will be analyzed. Preliminary analysis of a single gravity assist at Venus or Mars didn't produce any trajectory with a total ΔV lower than the quick transfer (0-revolution) solutions discussed in the following sections. At this time the multiple revolution solutions to Lambert's problem have not yielded any useful result. With a maximum notice of 7 years, missions with multiple gravity assists such at Galileo's Venus-Earth-Earth gravity assist would likely not allow for interception before the Earth-Apophis collision in 2036. Future version of the software will also include the ability to search for phasing and resonant orbits to help determine additional launch windows and trajectories.

The mission design software was used to determine the so-called porkchop plot, a plot of minimum ΔV, search for launch windows, and to find the optimal trajectory for each window. For the rendezvous mission analysis, the maximum mission length allowed is 500 days. Increasing the mission length doesn't result in additional launch windows. The maximum mission length for the intercept mission used in this study is 1000 days. This allows for an interception mission to be carried out almost anytime during the 7 year analysis.

Also, a departure from a geostationary transfer orbit, as assumed for the IPBM system, is assumed for all of the following analyses.

Rendezvous Mission Analysis

A contour plot of the total ΔV for the time-of-flight (the number of days to rendezvous) versus launch dates is shown in Figure 1. This contour plot is similar to the porkchop plot seen throughout literature. Careful analysis of this plot indicates that launch windows will be available until in the 2029 and the early 2030's, as well as several times after 2035. The main constraint for launch windows after 2035 is that interception must occur at least 15 days prior to the Earth-Apophis collision.[2]

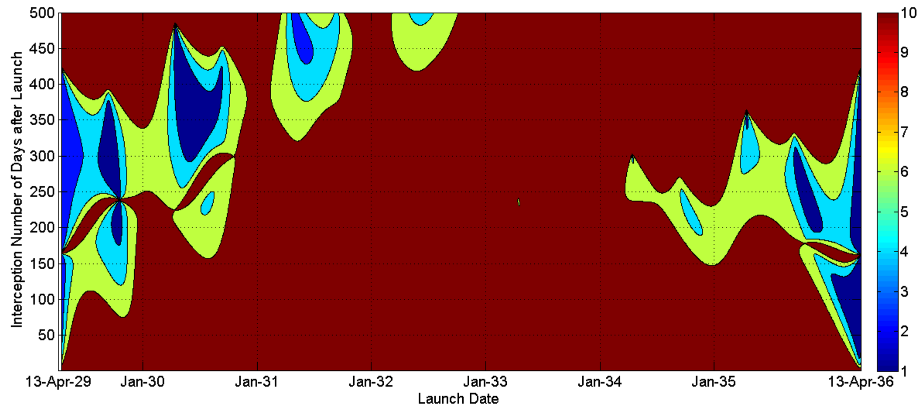

Figure 1: Total ΔV contour plot of the time-of-flight versus launch dates from 4/13/2029 to 4/13/2036 for the rendezvous mission.

Launch windows can be determined by examining Figure 2, which is a plot of minimum total ΔV versus launch date from 2029 to 2036. To determine the launch windows, a maximum ΔV of 4 km/s was used. There are 4 possible launch windows from 2029 to 2031. After this time there are no feasible rendezvous launch windows prior to the Earth-Apophis collision in 2036. Information for the nominal departure date for each windows can be found in Table 2. This table shows the magnitudes and dates for the Earth departure and Apophis arrival burns as well as the orbital elements for each trajectory.

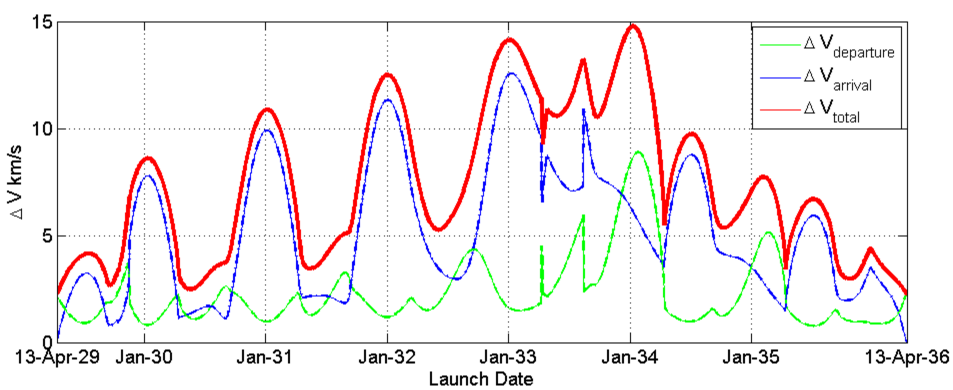

Figure 2: Minimum total ΔV versus launch date from 4/13/2029 to 4/13/2036.

Each separate launch window is shown in Figures 3, 4, 5, and 6 respectively. For each launch window plot, the Earth departure, Apophis arrival, and total ΔV's are provided. Launch windows range from 70-155 days in length, enough time to launch multiple IPBM's. Analysis of these 4 launch windows shows significant variations in the Earth departure (1-2.2 km/s) and Apophis arrival (0.1-2.15 km/s) maneuvers.

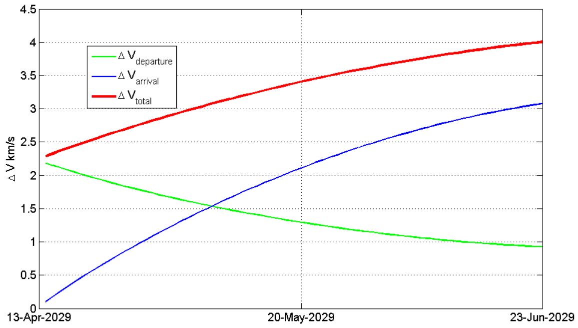

Figure 3: Launch window 1.

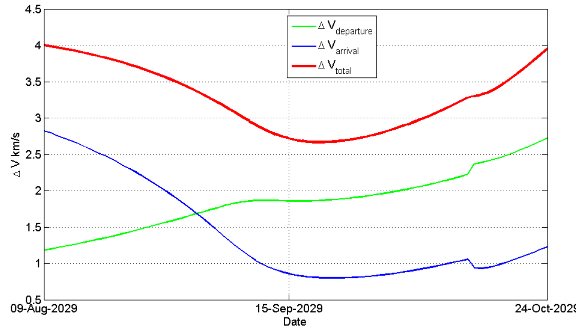

Figure 4: Launch window 2.

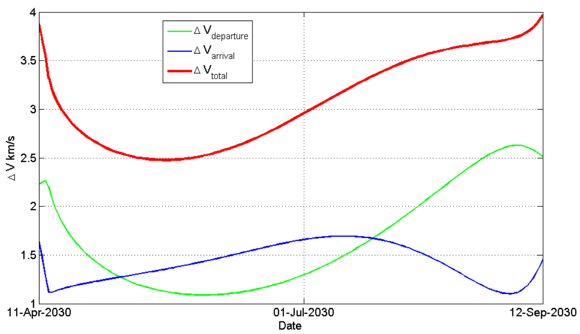

Figure 5: Launch window 3.

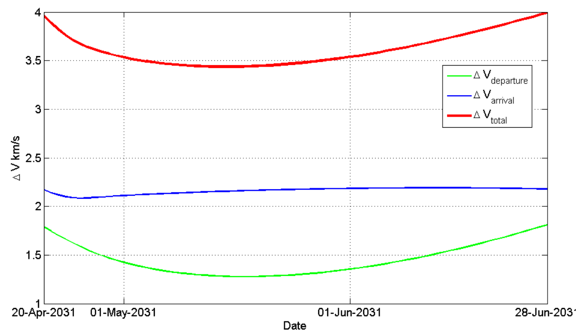

Figure 6: Launch window 4.

The arrival dates corresponding to the early launch windows are shown in Figure 7. There are no possible rendezvous launch dates after 2031. It is assumed that a decision to launch a deflection mission would not be made prior to mid-2031, which is the last of the early launch dates. For this reason "last minute" disruption mission will be discussed in a later section.

Interplanetary Trajectories

The transfer trajectories corresponding to the minimum total ΔV for each launch window are shown in Figures 8, 9, 10, and 11 respectively. It can be seen from each of the trajectory plot that the typical Sun-Apophis approach angles are in the 90-deg range. Further analysis of the Sun-Apophis approach or phase angles is needed for terminal-phase guidance analysis and design.

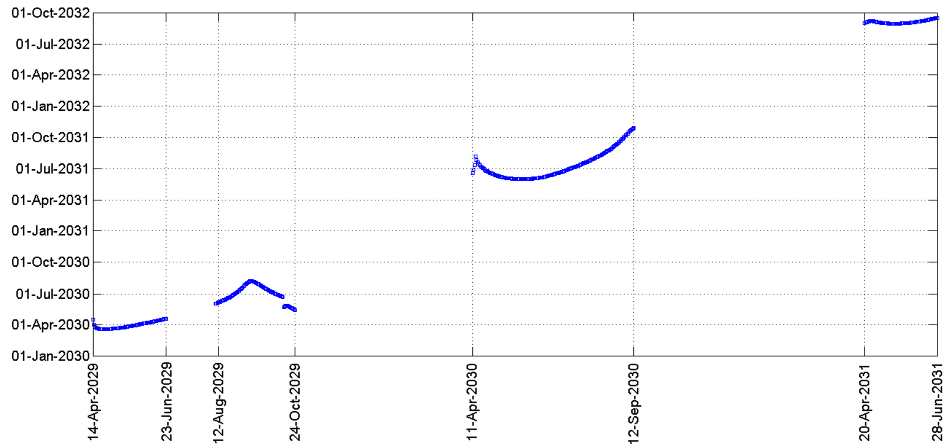

Figure 7: Arrival dates for the first four launch windows.

Table 2: Minimum ΔV transfer trajectory for each optimal launch date.

Mission Description	**Launch 1**	**Launch 2**	**Launch 3**	**Launch 4**
Earth Departure				
Departure Date	14-Apr-29	12-Sep-29	20-May-30	15-May-31
Departure C_3 (km^2/s^2)	33.247	25.399	7.201	11.538
Departure ΔV (km/s)	2.186	1.867	1.091	1.281
Transfer Orbit				
Semi-major axis (AU)	1.107	1.058	1.205	1.301
Eccentricity	0.188	0.155	0.161	0.223
Inclination (deg)	2.140	1.711	1.868	1.435
Ω (deg)	204.359	178.355	238.994	233.896
ω (deg)	69.943	100.649	7.475	358.253
Departure θ, deg	290.119	79.376	352.639	1.895
Arrival θ, deg	240.004	334.256	257.810	293.386
Apophis Arrival				
Arrival Date	16-Apr-30	26-Jul-30	2-Jun-31	30-Aug-32
Arrival ΔV (km/s)	0.102	0.801	1.460	2.156
Total ΔV	**2.289**	**2.668**	**2.552**	**3.436**
Launch Window Length (days)	**71**	**77**	**155**	**70**

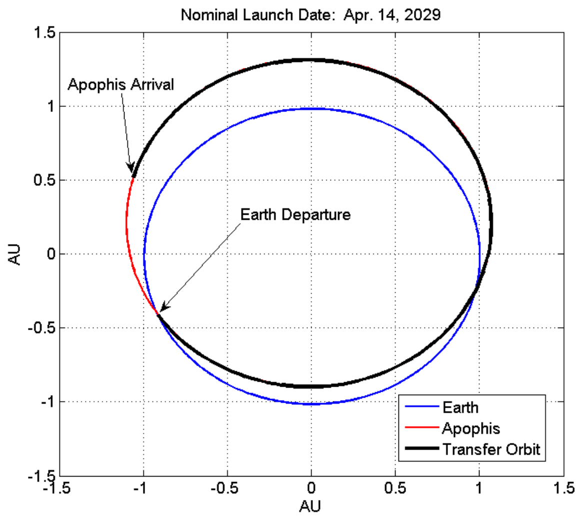

Figure 8: Nominal trajectory launch window 1. Trajectory shown in (X,Y) plane of J2000 coordinate system.

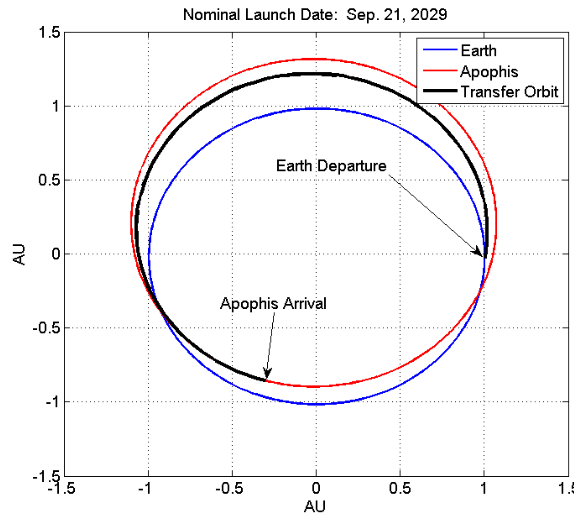

Figure 9: Nominal trajectory launch window 2. Trajectory shown in (X,Y) plane of J2000 coordinate system.

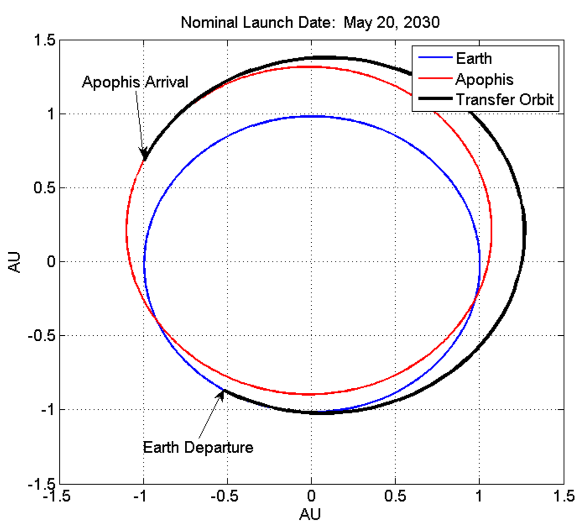

Figure 10: Nominal trajectory launch window 3. Trajectory shown in (X,Y) plane of J2000 coordinate system.

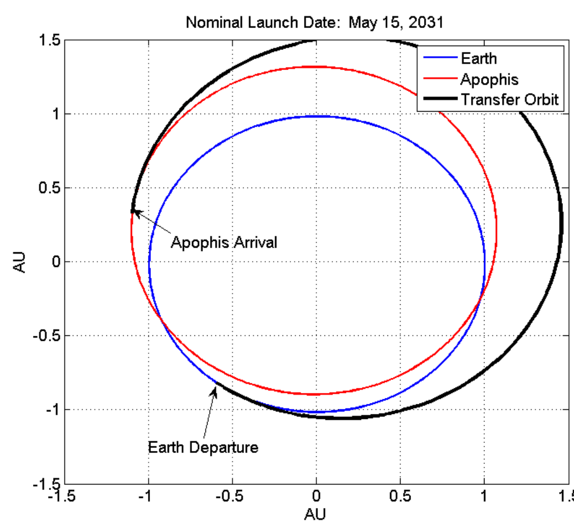

Figure 11: Nominal trajectory launch window 4. Trajectory shown in (X,Y) plane of J2000 coordinate system.

Late Launch Opportunities

As previously mentioned there are no feasible rendezvous launch windows after the end of 2031. It is likely that several years would be needed to develop and build a spacecraft such as the proposed IPBM system. With the last rendezvous launch window ending a little more than 2 years after the 2029 close encounter, it is not be possible to launch a disruption mission requiring a rendezvous maneuver at Apophis. For this reason a direct intercept mission will be analyzed as well. A total ΔV contour plot of the time-of-flight versus launch date is shown in Figure 12. From this plot it can be seen that an intercept mission with a maximum ΔV of 4 km/s is possible at almost any time from 2029-2036.

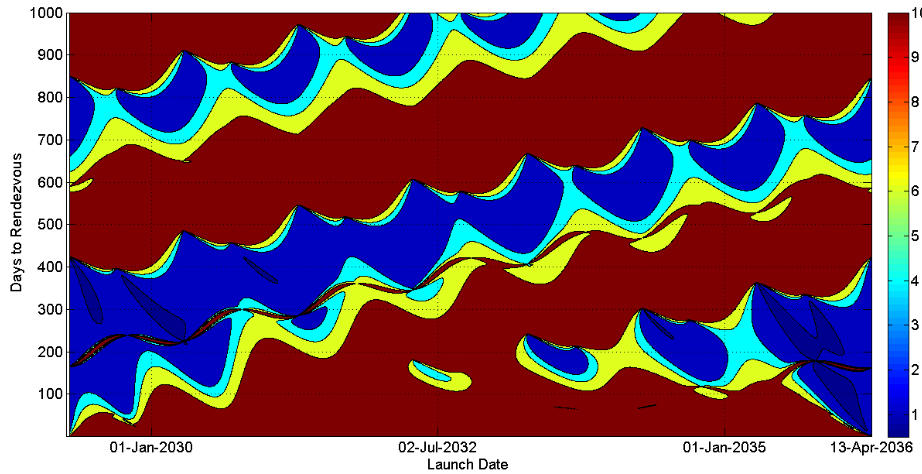

Figure 12: Total ΔV contour plot of the time-of-flight versus launch dates from 4/13/2029 to 4/13/2036 for the interception mission.

Recent research[2] has shown that it may be possible to prevent all but a few percent of the material from an asteroid from colliding with the Earth by utilizing a nuclear subsurface explosion. However, such a disruption mission requires a rendezvous with Apophis to provide an accpetable penetrator velocity of 300 m/s. With this information the objective of this section is to determine the feasibility of a last minute rendezvous mission launched anywhere from 2-3 years to as little as 20-30 days prior the the 2036 collision. To do this requires further analysis of Figure 12. An expanded version of Figure 12 with which the feasibility of late launches can be determined is shown in Figure. 13.

Careful analysis of Figure 13 shows that late launch dates ranging from 3/28/2035 to 3/19/2036 are feasible for the interception mission. The last feasible launch date is 25 days prior to the fictitious Earth-Apophis close encounter. Using the IPBM system architecture, it may be feasible to launch a last minute intercept mission to disrupt an NEO similar to Apophis. No limits on the arrival velocity have been imposed in Figure 12 and 13. Unfortunately, the penetrators maximum impact velocity is approximately 300 m/s, which means the the late intercept mission concept may not be a viable option. In this situation either a nuclear surface explosion (contact burst) or high-speed penetrator (¿5 km/sec impact velocity) must be developed and employed. An example of late intercept trajectory is shown in Figure 14, with all necessary mission data shown in Table 3.

To determine the feasibility of the late interception mission, further information is needed on an impact penetrator used for the nuclear subsurface explosion. In general, the later the intercept launch occurs the higher the arrival V_∞ at Apophis. Intercept launch dates in the 2034-2035 range generally have an intercept velocity (at Apophis) in the 5-6 km/s range.

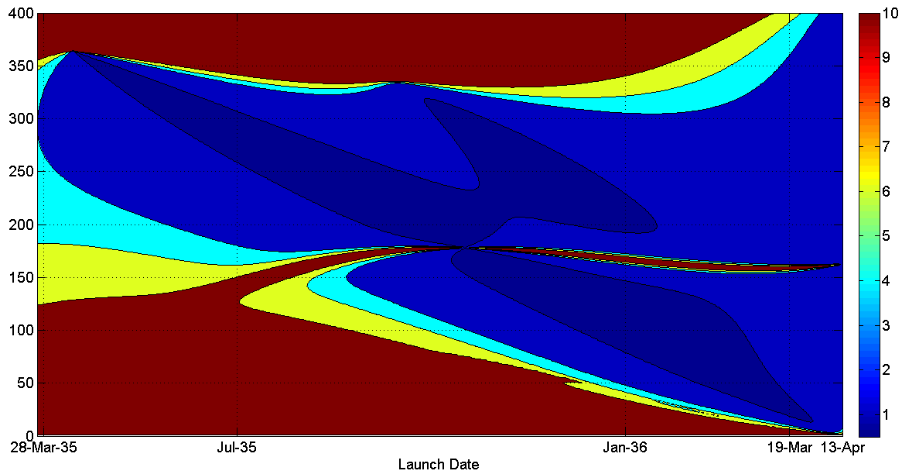

Figure 13: Total ΔV contour plot of the flight-of-time versus launch dates from 4/13/2029 to 4/13/2036 for the interception mission.

Table 3: Mission description of a last minute interception mission

Example Trajectory	
Departure Date	26-Feb-36
Arrival Date	19-Mar-36
Total ΔV, km/s	2.830
Semi-major axis, AU	1.714
Eccentricity	0.441
Inclination, deg	2.032
Ω, deg	336.944
ω, deg	153.860
Departure θ, deg	26.163
Arrival θ, deg	50.231

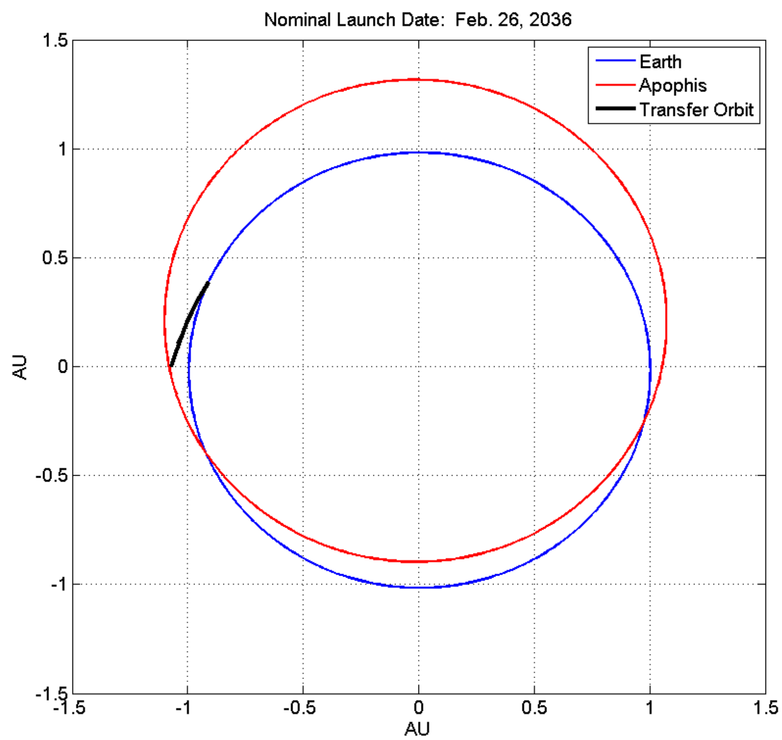

Figure 14: Trajectory for a late intercept mission. Trajectory shown in (X,Y)-plane of J2000 coordinate system.

CONCLUSION

The software necessary to determine launch windows and trajectories for rendezvous or interception missions has been developed. Analysis of a fictive post-2029 mission for asteroid Apophis has resulted in 4 possible launch windows. However, these launch windows all occur 4-5 years prior to the Earth impact in 2036. The direct interception mission analysis has also been performed to determine the feasibility of late interception missions. It has been shown that interception missions, without arrival velocity constraints, are possible up to 25 days prior to impact. However, such a late launch will not be a viable option because a nuclear subsurface explosion requires a penetrator with an impact speed of 300 m/s. For this reason the late launch interceptions missions may not be of any practical use for the subsurface nuclear detonation, unless a high speed (¿5 km/s) penetrator becomes available in the future.

Further analysis is needed, possibly using a combination of phasing orbits, multiple revolution orbits, and resonant orbits to overcome this problem, in order to determine viable rendezvous launch dates in the 2034-2035 range. This is needed because of the arrival speed constraint of 300 m/s of the nuclear explosive device. The software will be further developed to include these abilities to search for additional launch dates with more favorable asteroid arrival velocities. In addition, more information on the penetrator requirement is needed to determine the feasibility of late interception launch windows.

ACKNOWLEDGMENTS

This work was supported by the Iowa Space Grant Consortium (ISGC) through a research grant to the ADRC at Iowa State University. The authors would like to thank Dr. Willam Byrd (Director, ISGC) for his interest and support of this research work.

APPENDIX A: SOLUTION TO LAMBERT'S PROBLEM

The purpose of the appendices are to lay the ground work necessary to build a computational tool to determine the trajectories and information presented in this paper. Several solutions to Lambert's problem have been tested to determine the most efficient solution (Appendix A). The transformation from orbital elements to state vectors will be covered as well as the inverse transformation from the state vectors to the orbital elements. Ephemeris data will also be needed to calculate the positions of the Earth and each asteroid that is tested. A linear solution to determine Earth's orbital elements[5] will be covered in Appendix B. Ephemeris data for each asteroid tested was obtained using NASA's HORIZONS System. A solution to Kepler's time equations using a Newton method is included in Appendix B as well.

For any given two position vectors and the time-of-flight (TOF), the orbit determination problem is called Lambert's problem. For this reason Lambert's problem is well suited for an initial orbit-determination and searching technique. In Lambert's problem the two position vectors and the TOF are known, but the orbit connecting those two points is unknown. The classical Kepler problem is used to determine a position as a function of time where the initial position and velocity vectors are known. Various solutions so Lambert's problem can be found in the literature.[5,6,7,8] While many solutions methods have been proposed, two commonly used methods are employed, namely the classical universal variable method[5,7,8] and Battin's more recent approach using an alternate geometric transformation than Gauss' original method.[6,7] The search for launch dates often requires Lambert's problem to be solved thousands, often millions of time. For this reason the object of this section is to determine the most efficient method to solve Lambert's problem.

Lambert Problem Definition

In Lambert's problem the initial and final radius vectors and time-of-flight are given respectively as, $\vec{r}_0$, $\vec{r}$, and Δt. The magnitudes of $\vec{r}_0$ and $\vec{r}$ are defined as r_0 and r. The following parameters are also needed for the post processing for each solution method.

$$c = \mid \vec{r} - \vec{r}_0 \mid \tag{1}$$

$$s = \frac{r_0 + r + c}{2} \tag{2}$$

A method to determine the transfer angle $\Delta\theta$ without quadrant ambiguity is described below.[5] The transfer angle $\Delta\theta$ for a prograde orbit is determined as follows:

$$\Delta\theta = \begin{cases} \cos^{-1}\left(\dfrac{\vec{r}_0 \cdot \vec{r}}{r r_0}\right) & \text{if } (\vec{r}_0 \times \vec{r}) \geq 0 \\ 360^o - \cos^{-1}\left(\dfrac{\vec{r}_0 \cdot \vec{r}}{r r_0}\right) & \text{if } (\vec{r}_0 \times \vec{r}) < 0 \end{cases} \tag{3}$$

Similarly the transfer angle for a retrograde orbit is determined as:

$$\Delta\theta = \begin{cases} \cos^{-1}\left(\dfrac{\vec{r}_0 \cdot \vec{r}}{r r_0}\right) & \text{if } (\vec{r}_0 \times \vec{r}) < 0 \\ 360^o - \cos^{-1}\left(\dfrac{\vec{r}_0 \cdot \vec{r}}{r r_0}\right) & \text{if } (\vec{r}_0 \times \vec{r}) \geq 0 \end{cases} \tag{4}$$

Battin's Solution

Let's consider Battin's approach first proposed in the 1980's and later published in his book.[6] This solution is similar to the method used by Gauss, but moves the singularity from 180^o to 360^o and dramatically improves convergence when the θ is large. It should also be noted that Battin's solution works for all types of orbits (elliptical, parabolic, and hyperbolic) just as Gauss's original solution does. Throughout this section a brief outline of Battin's solution will be given. Any details left out can be found in Battin's book.[6]

Geometric Transformation of Orbit Lambert's problem states that the time-of-flight is a function of a, $r_0 + r$, and c. Therefore the orbit can be transformed to any shape desired as long as the semi-major axis a, r_0+r, and c are held constant. For Battin's formulation[6,9,10] the orbit is transformed such that the semi-major axis is perpendicular to the line from $\vec{r}_0$ to $\vec{r}$, which is c by definition. The geometry of the transformed ellipse is shown in Battin's book.[6] From here the equation for the pericenter radius, r_p, is given below. The details leading up to this function can be found in Ref. (6).

$$r_p = a(1 - e_0) = r_{0p} \sec^2 \frac{1}{4}\left(E - E_0\right) \tag{5}$$

where e_0 is the eccentricity of the transformed orbit, while r_{0p} is the mean point of the parabolic orbit from P_0 to P.[6] The mean point of the parabolic radius r_{0p} is also given by:[6]

$$r_{0p} = \sqrt{r_0 r}\left[\cos^2\left(\frac{\Delta\theta}{4}\right) + \tan^2\left(2w\right)\right] \tag{6}$$

The value of $\tan^2(2w)$ is needed for the calculation, so w itself will not be defined. Next Kepler's time of flight equation can be transformed into a cubic equation, which must then be solved. First y will be defined as:

$$y^2 \equiv \frac{m}{(\ell + x)(1 + x)} \tag{7}$$

Kepler's time-of-flight equation can now be represented by the following cubic equation.

$$y^3 - y^2 - \frac{m}{2x}\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}} - \frac{1}{1+x}\right) = 0 \tag{8}$$

In the above equation l, m, and x are always positive and are defined below. To calculate l, the following two equations, dependent only on the geometry of the problem, are also necessary.

$$\epsilon = \frac{r - r_0}{r_0} \tag{9}$$

$$\tan^2(2w) = \frac{\frac{\epsilon^2}{4}}{\sqrt{\frac{r}{r_0}} + \frac{r}{r_0}\left(2 + \sqrt{\frac{r}{r_0}}\right)} \tag{10}$$

Using Eqs. (9) and (10), ℓ can be calculated as follows.

$$\ell = \frac{\sin^2\left(\frac{\Delta\theta}{4}\right) + \tan^2(2w)}{\sin^2\left(\frac{\Delta\theta}{4}\right) + \tan^2(2w) + \cos\left(\frac{\Delta\theta}{2}\right)} \qquad 0^o < \Delta\theta \leq 180^o \tag{11}$$

$$\ell = \frac{\cos^2\left(\frac{\Delta\theta}{4}\right) + \tan^2(2w) - \cos\left(\frac{\Delta\theta}{2}\right)}{\cos^2\left(\frac{\Delta\theta}{4}\right) + \tan^2(2w)} \qquad 180^o < \Delta \leq << 360^o \tag{12}$$

$$m = \frac{\mu \Delta t^2}{8 r_{op}^3} \tag{13}$$

$$x = \sqrt{\left(\frac{1-\ell}{2}\right)^2 + \frac{m}{y^2}} - \frac{1+\ell}{2} \tag{14}$$

Initial conditions for x that guarantee convergence are:

$$x_0 = \begin{cases} 0 & \text{parabola, hyperbola} \\ \ell & \text{ellipse} \end{cases} \tag{15}$$

The cubic function (Eq. (8)) can now be solved using the following sequential substitution method:

1. An initial estimation of x is given by Eq. (15).
2. Calculate all the values needed for the cubic from Eqs. (9), (10), (11) or (12), and (13).
3. Solve the cubic Eq. (8) for y.
4. Use Eq. (14) to determine a new value for x.
5. Repeat the above 3 steps until x stops changing.

We now have a nearly complete solution algorithm for Lambert's problem, although solving the cubic function is not a trivial matter. Battin[6] next develops a method to flatten the cubic function to improve the convergence rate, as well as a method using continued fractions to determine the largest real positive root of the cubic function. The following section gives the equations necessary to find the flattening parameters and solve the cubic function. Again, a much more detailed derivation can be found in Ref. (6).

Use of Free Parameter to Flatten Cubic Function The cubic equation, Eq. (8), from the previous section can now be represented in terms of the flattening parameter functions, h_1 and h_2, as follows:

$$y^3 - (1 + h_1)y^2 - h_2 = 0 \tag{16}$$

The flattening parameters can be represented in terms of x, l, and m as

$$h_1 = \frac{(l + x)^2(1 + 3x + \xi)}{(1 + 2x + l)[4x + \xi(3 + x)]} \tag{17}$$

$$h_2 = \frac{m(x - l + \xi)}{(1 + 2x + l)[4x + \xi(3 + x)]} \tag{18}$$

The function $\xi(x)$ needed for the calculation of h_1 and h_2 is calculated from the continued fraction as follows:

$$\xi(x) = \cfrac{8(\sqrt{1+x}+1)}{3 + \cfrac{1}{5 + \eta + \cfrac{\frac{9}{7}\eta}{1 + \cfrac{\frac{16}{63}\eta}{1 + \cfrac{\frac{25}{99}\eta}{1 + \cfrac{\frac{36}{143}\eta}{1 + \ddots}}}}}} \tag{19}$$

where η is defined as

$$\eta = \frac{x}{(\sqrt{1+x}+1)^2} \qquad \text{where} \qquad -1 < \eta < 1 \tag{20}$$

Solving the Cubic Function In this section a method to determine the largest real root of Eq. (16) is outlined. Using this method a successive algorithm can be used ultimately to determine the f and g functions, given by Ref. (7,8). Using the Lagrangian f and g functions the initial and final velocity vectors are then obtained. The first step is to calculate B and u as follows:

$$B = \frac{27h_2}{4(1 + h_1)^3} \tag{21}$$

$$u = \frac{B}{2(\sqrt{1+B}+1)} \tag{22}$$

Also, $K(u)$ is the calculated from the continued fraction

$$K(u) = \cfrac{\frac{1}{3}}{1 + \cfrac{\frac{4}{27}u}{1 + \cfrac{\frac{8}{27}u}{1 + \cfrac{\frac{2}{9}u}{1 + \cfrac{\frac{22}{81}u}{1 + \ddots}}}}} \tag{23}$$

where the coefficients for the odd and even coefficients of u are generally obtained from the following two equations.

$$\gamma_{2n+1} = \frac{2(3n+2)(6n+1)}{9(4n+1)(4n+3)} \tag{24}$$

$$\gamma_{2n} = \frac{2(3n+1)(6n-1)}{9(4n-1)(4n+1)} \tag{25}$$

The largest positive real root for the cubic equation is calculated as

$$y = \frac{1+h_1}{3}\left(2 + \frac{\sqrt{1+B}}{1+2u(K^2(u))}\right) \tag{26}$$

The cubic function (Eq. (16)) can now be solved using the following sequential substitution method.

1. An initial estimation of x is given by Eq. (15).
2. Calculate all the values needed for the flattening parameters Eqs.(17) and (18) from Eqs. (9), (10), (11) or (12), and (13).
3. Calculate $\eta(x)$ from Eqs. (19) and (20).
4. Calculate $K(u)$ using Eqs. (21), (22), and (23)
5. Calculate the solution for the cubic function using Eq. (26) for y.
6. Use Eq. (14) to determine a new value for x.
7. Repeat the above 5 steps until x stops changing.

The next step is to determine the semi-major axis of the orbit. If the semi-major axis is positive,the orbit is elliptical, and the initial and final velocity vectors can be calculated as follows. The hyperbolic and parabolic velocity vectors are calculated in a similar manner.[7]

$$a = \frac{\mu(\Delta t)^2}{16 r_{op}^2 x y^2} \tag{27}$$

Determining the Lagrange Coefficients for Elliptical Orbits The Lagrange coefficients for the parabolic orbit case can be calculated from the following set of equations.

$$\beta_e = 2\sin^{-1}\sqrt{\frac{s-c}{2a}} \tag{28}$$

$$\beta_e = -\beta_e \quad \text{If } \Delta\theta > \pi \tag{29}$$

$$a_{min} = \frac{s}{2} \tag{30}$$

$$t_{min} = \sqrt{\frac{a_{min}^3}{\mu}}(\pi - \beta_e + \sin\beta_e) \tag{31}$$

$$\alpha_e = 2\sin^{-1}\sqrt{\frac{s}{2a}} \tag{32}$$

$$\alpha_e = 2\pi - \alpha_e \quad \text{If } \Delta t > t_{min} \tag{33}$$

$$\Delta E = \alpha_e - \beta_e \tag{34}$$

The Lagrangian coefficients can be calculated as follows:

$$f = 1 - \frac{a}{r_0}(1 - \cos\Delta E) \tag{35}$$

$$g = \Delta t - \sqrt{\frac{a^3}{\mu}}(\Delta E - \sin\Delta E) \tag{36}$$

$$\dot{g} = 1 - \frac{a}{r}(1 - \cos\Delta E) \tag{37}$$

With the Lagrangian coefficients now known, the initial and final velocity vectors can be found from the following equations.

$$\vec{v}_0 = \frac{\vec{r} - f\vec{r}_o}{g} \tag{38}$$

$$\vec{v} = \frac{\dot{r}\vec{r} - \vec{r}_o}{g} \tag{39}$$

The f and g functions can also be determined for the hyperbolic case.[7]

This concludes the solutions to Lambert's problem using Battin's method.

Computational Efficiency for Each Solution Method

In this section an overview of the performance obtained from various Lambert problem solutions will be discussed. The three methods covered in this section are Battin's solution (discussed above), a universal variable using a Newton method, similar to BMW[8] and Curtiss,[5] and lastly a universal variable solution using the bisection method, similar to Vallado.[7]

Table 4: Information on the computer used for the computational efficiency study.

Model	Dell T3500 Workstation
Operating System	Windows Vista Enterprise 64 bit
Processor	2 x Intel(R) Xeon(R) W3520 2.67 GHz
Memory	6.00 GB 1066 Mhz DDR3
Matlab	R2009b-64 bit

For all calculations Matlab R2009b (64 bit version) was used. A compile environment would likely increase the general speed of the program, however the program is being written so the final version can be easily used and modified by others at the ADRC. Table 4 has all the information relevant to the computer used for these calculations. The computer used was a dual quad core Xeon processor (8 total cores) running Windows Vista Enterprise.

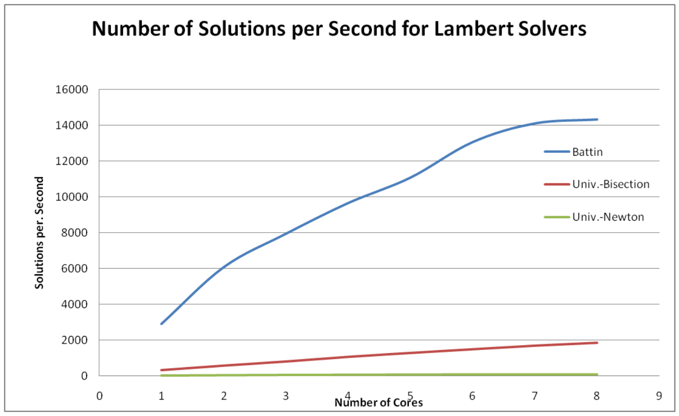

Figure 15: Plot of number of solutions per second versus number of cores for various Lambert solution methods.

In each case the program was ran for an example case that is representative of cases that the program would be likely to be used for. In this case it was ran to compute a porkchop plot for and asteroid rendezvous mission. A plot of the number of solutions per second for each of the three solutions is given in Fig. 15. Analysis of this plot indicates that the Battin solution is significantly more efficient than either the Newton or bisection solutions. In the same way the Newton universal variable solution is much slower than the bisection method. In general the Newton method would be expected to be more efficient than a bisection method. In this case it is likely that the Newton method doesn't always converge and runs through every loop iteration allowed by the program. On average the Battin Solution is 8.94 times faster than the bisection method and 158.66 times faster than Newton solution method. The maximum number of solutions obtain using all 8 cores (maximum in every case) for the Battin, Newton, and bisection method are 14317, 1867, and 174 respectively.

APPENDIX B: MAIN FUNCTIONS FOR THE SEARCH PROGRAM

Julian Date

One of the first steps to determine the Earth's, and any other Planet's, orbital elements is to determine the Julian date. Ephemeris data is often given as a function of time, which is represented by the Julian date. The Julian date is number of days that has passed since Jan. 1, 4713 BC 12:00 at Greenwich. The conversion to Julian date from the Gregorian calendar is given by the following equations:[5]

$$JD = J_0 + \frac{UT}{24}$$

where UT is defined to be the number of hours past midnight at Greenwich in decimal form and J_0 is calculated from:

$$J_0 = 367y - INT\left[\frac{7\left[y + INT\left(\frac{m+9}{12}\right)\right]}{4}\right] + INT\left(\frac{275m}{9}\right) + d + 1721013.5$$

where INT is taken to be the integer value of the number without rounding. The ranges for y, m, and d are:

$$1901 \le y \le 2099$$

$$1 \le m \le 12$$

$$1 \le d \le 31$$

Ephemeris Data

To calculate the ephemeris data the Julian date must first be calculated.[5] Once the Julian date is determined, the planetary orbital elements can be calculated from ephemeris tables. The method presented in this section has linear drift rates for each orbital elements, as well as initial values. In this way the orbital elements of each planet can be determined as a function of Julian date. Table (5) shows the drift rates and initial values for Earth.[5] A table with all the information for the other 7 planets and Pluto can be found in Ref. (5).

Table 5: Table of Earths orbital elements and drift rates used to generate ephemeris data.

a, AU $\dot{a}$, AU/Cy*	e $\dot{e}$, 1/Cy	i, deg $\dot{i}$,″†/Cy	Ω , deg $\dot{\Omega}$, ″/Cy	$\tilde{\omega}$, deg $\dot{\tilde{\omega}}$, ″/Cy	L, deg $\dot{L}$, ″/Cy
1.00000011	0.01671022	0.00005	-11.26064	102.94719	100.46435
-0.00000005	-0.00003804	-46.94	-18228.25	1198.28	129597740.6

* where Cy is a century, which is 36525 days.
† where ″ is symbol for an arcsecond, which 1/3600th of a degree.

The orbital elements are calculated in the following manner. If X is defined to be any of the 6 orbital elements, then the orbital elements can be calculated in the following manner.

$$X = X_0 + \dot{X}T_0 \tag{40}$$

where T_0, the number of Julian centuries between J2000 and the date, is calculated from the Julian date.

$$T_0 = \frac{JD - 2451545}{36525} \tag{41}$$

$$\omega = \tilde{\omega} - \Omega \tag{42}$$

$$M = L - \tilde{\omega} \tag{43}$$

From the mean anomaly (M) and the eccentricity (e), the eccentric anomaly (E) can be found. Then, the true anomaly (θ) easily can be found. Finding the eccentric angle from M and e requires the Newton iteration method to solve Kepler's time equation, which is described in Curtis.[5]

Obtaining the State Vector from Orbital Elements

With all six orbital elements found from the ephemeris routine, the state vector can be calculated. The radius vector is needed for the inputs to Lambert's problem, as well for plotting trajectories. The velocity vectors are also needed to calculated ΔV's when the each Lambert solution is obtained. The function resulting from this section will be useful in many other situations as well. The first step is to calculate the radius and velocity vectors in the perifocal frame.[5,7]

$$\vec{r}_{pqw} = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta}\left\{\begin{array}{l}\cos\theta\\ \sin\theta\\ 0\end{array}\right\} \tag{44}$$

$$\vec{v}_{pqw} = \frac{h}{\mu}\left\{\begin{array}{l}-\sin\theta\\ e+\cos\theta\\ 0\end{array}\right\} \tag{45}$$

Where the specific angular moment, h, is given by

$$h = \sqrt{a(1-e^2)\mu} \tag{46}$$

The transformation matrix from the perifocal frame to the body centered frame is given by.[3]

$$Q_{pqw\rightarrow ijk} = \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin w\sin i & \cos w\sin i & \cos i \end{bmatrix} \tag{47}$$

The radius and velocity vectors in the body centered frame are then transformed by

$$\vec{r}_{ijk} = Q\vec{r}_{pqw} \tag{48}$$

$$\vec{v}_{ijk} = Q\vec{v}_{pqw} \tag{49}$$

Obtaining Orbital Elements from State Vector

In this section a method to obtain the six classical orbital elements, a, e, i, Ω , ω, and θ, from the state vector will be discussed. The sixth orbital element is actually, t_p, but this can be obtained directly from θ. This conversion is necessary to propagate orbits and determine the orbital elements after the solution to Lambert's problem has been obtained. The first step is to determine the specific angular momentum vector as follows:

$$r = |\vec{r}| \tag{50}$$

$$v = |\vec{v}| \tag{51}$$

$$\vec{h} = \vec{r} \times \vec{v} \tag{52}$$

$$h = |\vec{h}| \tag{53}$$

The eccentricity vector and orbit eccentricity can be found as

$$\vec{e} = \frac{1}{\mu}\left[\vec{v} \times \vec{h} - \mu\frac{\vec{r}}{r}\right] \tag{54}$$

$$e = |\vec{e}| \tag{55}$$

With the angular momentum and eccentricity now known, the orbit's semi-major axis can be found as

$$a = \frac{h^2}{\mu}\frac{1}{1-e^2} \tag{56}$$

The inclination of the orbit can be obtained from the angular momentum vector as the inverse cosine of the z component of the angular momentum vector divided by the magnitude of the angular moment. By definition the inclination must be between 0^o to 360^o, so there won't be any quadrant ambiguities.

$$i = \cos^{-1}\left(\frac{h_z}{h}\right) \tag{57}$$

The node line vector is required to obtain both the right ascension of the ascending node, and the argument of perigee of the orbit. The node line is calculated from

$$\vec{N} = \vec{K} \times \vec{h} \tag{58}$$

$$N = |\vec{N}| \tag{59}$$

The longitude of the ascending node Ω is found from $\Omega = cos^{-1}\frac{N_x}{N}$, with the proper quadrant, as

$$\Omega = \begin{cases} \cos^{-1}\left(\frac{N_x}{N}\right) & N_y \geq 0 \\ 360^o - \cos^{-1}\left(\frac{N_x}{N}\right) & N_y < 0 \end{cases} \tag{60}$$

The argument of perigee and the true anomaly can be found similarly as

$$\omega = \begin{cases} \cos^{-1}\left(\frac{\vec{N}\cdot\vec{e}}{Ne}\right) & e_z \geq 0 \\ 360^o - \cos^{-1}\left(\frac{\vec{N}\cdot\vec{e}}{Ne}\right) & e_z < 0 \end{cases} \tag{61}$$

$$\theta = \begin{cases} \cos^{-1}\left(\frac{\vec{e}\cdot\vec{r}}{er}\right) & \vec{r}\cdot\vec{v} \geq 0 \\ 360^o - \cos^{-1}\left(\frac{\vec{e}\cdot\vec{r}}{er}\right) & \vec{r}\cdot\vec{v} < 0 \end{cases} \tag{62}$$

Solution of Kepler's Time Equation

For the ephemeris routine above a solution to Kepler's time equation is required. In this case both mean anomaly angle, M, and and the eccentricity, e, of the orbit are known parameters. A simple solution using Newton's method is presented in this section.[5]

$$M = E - e\sin E \tag{63}$$

The first step for the Newton iteration process is to select and initial value for E.

$$E_0 = \begin{cases} M + \frac{e}{2} & \text{if} M < \pi \\ M - \frac{e}{2} & \text{if} M \geq \pi \end{cases} \tag{64}$$

The Newton process repeats the following two functions until E stops changing within a specified tolerance.

$$\text{error} = \frac{E_i - e\sin E_i - M}{1 - e\cos E_i} \tag{65}$$

$$E_{i+1} = E_i - \text{error} \tag{66}$$

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